MA 161: Lesson 27
Optimization Problems - Part II (4.5)

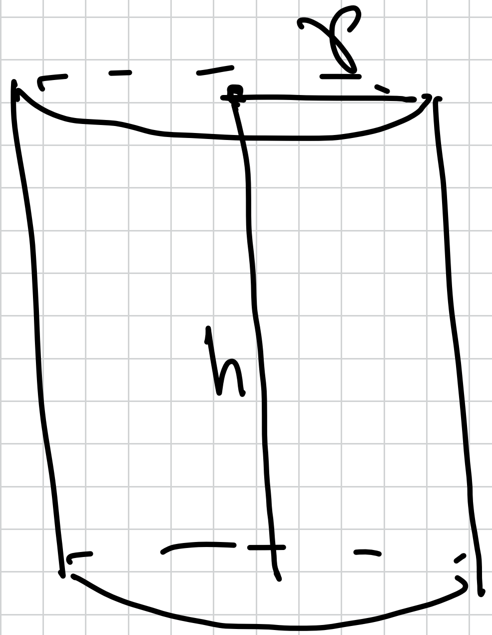
Strategy for Optimization Problems:

1) Read, sketch, label
2) identify: Objective
   Constraint
3) Use constraint to write objective as a function of 1-variable
4) take derivative, find critical points
   use 1st or 2nd derivative test to find max/min

Office Hours

M, W, F: 2:45pm - 4:15pm

An aluminum can is to contain $54 in^3$ of cola. What are the dimensions of such a can using the least amount of aluminum.



Constraint: Volume $= \pi r^2 h = 54$

Objective: Use least amount of aluminium

on the curved surface, Top, Bottom

$2\pi r h$ $\quad \pi r^2 \quad \pi r^2$

Obj: min of $A = 2\pi r h + 2\pi r^2$

$\pi r^2 h = 54 \rightsquigarrow h = \dfrac{54}{\pi r^2}$

Obj: $A(r) = 2\pi r \cdot \dfrac{54}{\pi r^2} + 2\pi r^2 = \dfrac{108}{r} + 2\pi r^2$

Find the minimum value of $A(r) = \frac{108}{r} + 2\pi r^2$, $r > 0$.

$$A'(r) = \frac{-108}{r^2} + 4\pi r = 0 \Leftrightarrow \frac{108}{4\pi} = r^3 \Rightarrow r^3 = \frac{27}{\pi}$$

$$r = \frac{3}{\pi^{1/3}}$$

Verify $r = \frac{3}{\pi^{1/3}}$ is minimum.

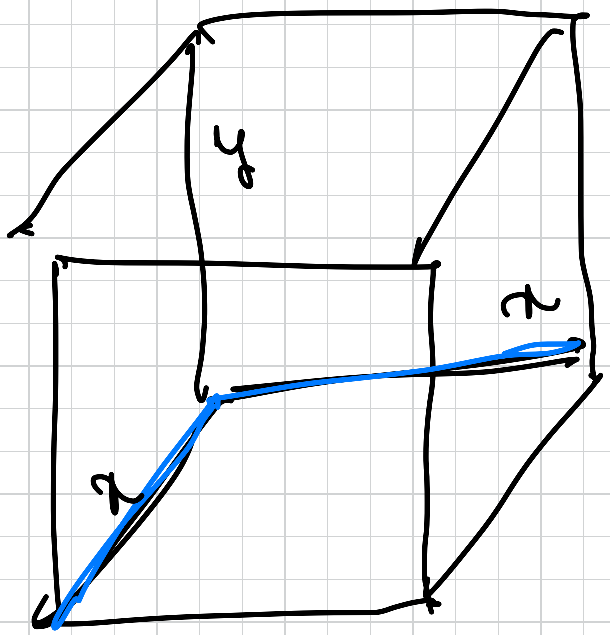
$$A''(r) = \frac{216}{r^3} + 4\pi > 0 \text{ for } r = \frac{3}{\pi^{1/3}}$$

$+ \quad +$

Least Amount of aluminium to Build a Can containing $54\ in^3$ of cola is when $r = \frac{3}{\pi^{1/3}}$

$$h = \frac{54}{\pi\left(3/\pi^{1/3}\right)^2} = \frac{6}{\pi^{1/3}}$$

What are the dimensions of the largest storage box with square base and open top, built from 1200 cm^2 of cardboard.



Objective: max Volume:

$$V = x^2 y$$

Constraint: Amount of Cardboad = 1200 cm^3

Surface Area

$\underbrace{\text{Base}}_{x^2} + 4 \underbrace{\text{Sides}}_{xy}$

$$x^2 + 4xy = 1200$$

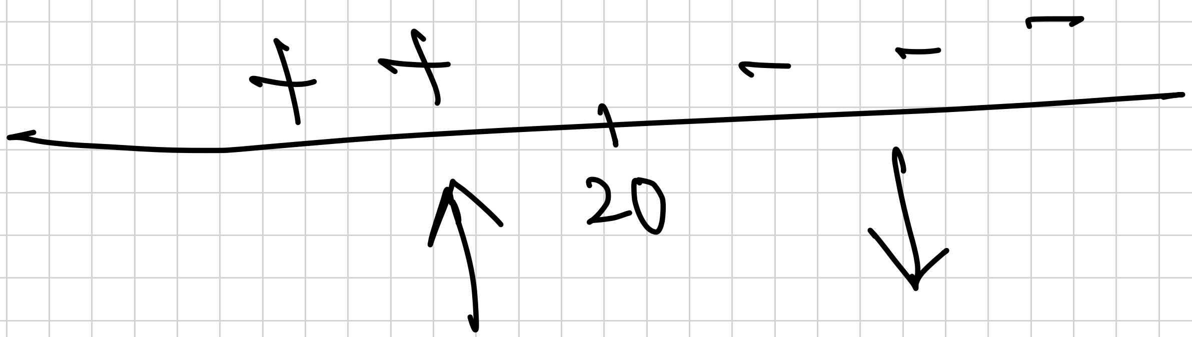
$$y = \frac{1200 - x^2}{4x} \rightsquigarrow V = x^2 y = x^2 \left[\frac{1200 - x^2}{4x} \right] = 300x - \frac{x^3}{4}.$$

Find max of $V(x) = 300x - \frac{1}{4}x^3$, $x > 0$

Critical: $V'(x) = 300 - \frac{3x^2}{4} = 0 \rightsquigarrow \frac{3x^2}{4} = 300 \Rightarrow x^2 = 400$

$x = 20$ or -20

1st Derivative test

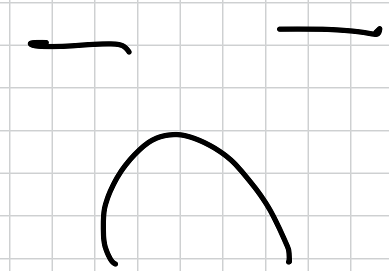


max at $x = 20$

2nd Derivative Test

$V''(x) = \frac{-6x}{4}$

$V''(20) = \frac{-6(20)}{4} < 0$



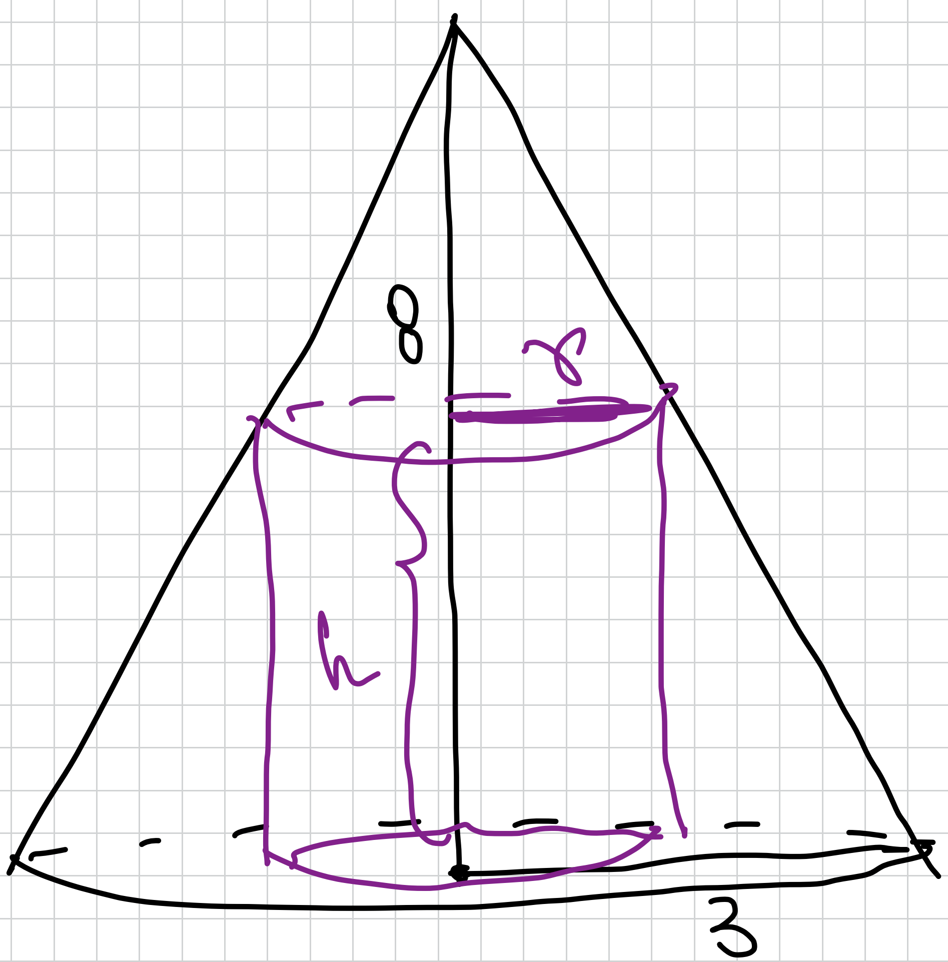
$\Rightarrow x = 20$ is max.

max. Volume $= 300(20) - \frac{1}{4}(20)^3 = 6000 - 2000 = 4000\ cm^3$.

Dimensions: $x = 20$ cm

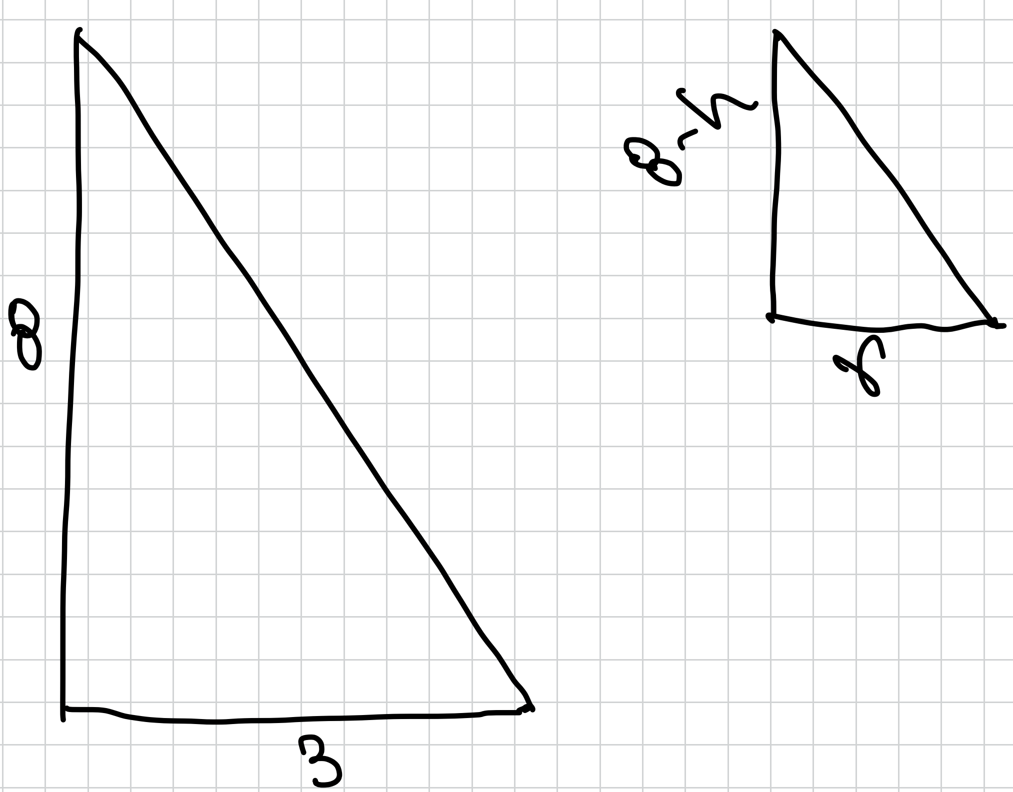
$y = \frac{1200 - (20)^2}{4(20)} = 10$ cm.

Find the volume of the largest cylinder that can be inscribed inside a cone of radius 3 in and height 8 in.



Objective: max $V = \pi r^2 h$

Constraint: Cylinder is inside cone. , $0 \le r \le 3$
$0 \le h \le 8$



Similar triangles

$$\frac{r}{3} = \frac{8-h}{8} \Rightarrow r = \frac{3}{8}(8-h)$$

$$V = \pi r^2 h = \pi\left(\frac{9}{64}\right)(8-h)^2 \cdot h$$

Find max. of $V(h) = \frac{9\pi}{64}(8-h)^2 \cdot h, \quad 0 \le h \le 8$.

$$V'(h) = \frac{9\pi}{64}\left[-2(8-h)\cdot h + (8-h)^2\right] = \frac{9\pi}{64}(8-h)(-2h+8-h)$$

$$= \frac{9\pi}{64}(8-h)(8-3h) = 0$$

$h = 8$ or $h = 8/3$

end point

$V(0) = 0$

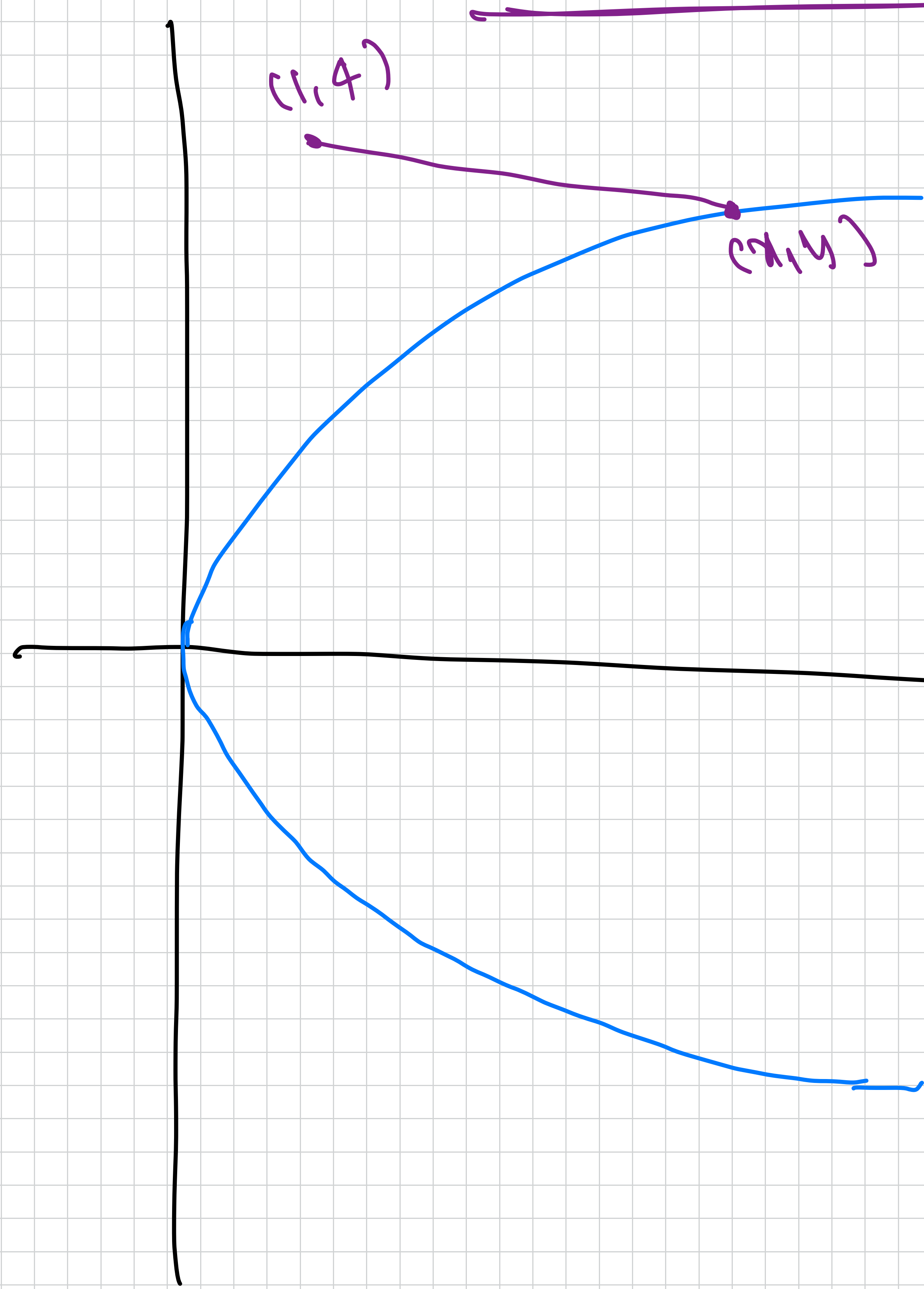
$V(8) = 0$

$$V(8/3) = \frac{9\pi}{64}\left(8-\frac{8}{3}\right)^2 \cdot \frac{8}{3} = \frac{9\pi}{64}\cdot\frac{256}{9}\cdot\frac{8}{3} > 0$$

max. volume of cylinder when $h = 8/3$ in

$r = \frac{3}{8}\left(8-\frac{8}{3}\right) = 2$ in.

What is the minimum distance from the point (1,4) to the curve $y^2 = 2x$?



objective: distance b/w (1,4) & (x,y) is minimum

min $\sqrt{(x-1)^2 + (y-4)^2}$

Constraint: (x,y) is on $y^2 = 2x$

$x = \frac{y^2}{2}$

Obj: min of $\sqrt{\left(\frac{y^2}{2}-1\right)^2 + (y-4)^2}$

find min value of $d(y)=\sqrt{\left(\frac{y^2}{2}-1\right)^2+(y-4)^2}$, $-\infty<y<\infty$.

Trick! d is min when d^2 is minimum, look at

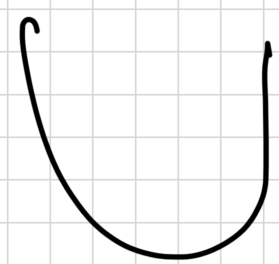
$$D(y)=d^2(y)=\left(\frac{y^2}{2}-1\right)^2+(y-4)^2$$

$$D'(y)=2\left(\frac{y^2}{2}-1\right)\cdot\left(\frac{1}{\cancel{2}}\cdot\cancel{2}y\right)+2(y-4)=(y^2-2)(y)+2y-8$$

$$=y^3-\cancel{2y}+\cancel{2y}-8=0$$

$$\rightsquigarrow\quad y^3=8\Rightarrow \underline{y=2}$$

$D''(y)=3y^2\geq 0$ + + $\rightsquigarrow$ min. when $y=2$.



$$x=\frac{y^2}{2}=\frac{2^2}{2}=2.$$

point on $y^2=2x$ closest to $(1,4)$

is $(2,2)$.